Convex optimization approach to signals with fast varying instantaneous frequency

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Objective

Goal

Represent the evolution of the instantaneous frequency of signals with fast varying instantaneous frequency

Steps

- Describe the type of signals we want to analyze
- Sort the desired properties of the ideal time-frequency representation ⇒ Construction of a functional to minimize
- Implement the minimization algorithm ⇒ Tycoon algorithm
- Compare the numerical results with other transforms

Motivations

Extracting proper features from the collected dataset is the first step toward data analysis.

Signal model

$$f(t) = \sum_{k=1}^{K} A_k(t) \cos(2\pi\phi_k(t))$$

where $A_k(t) > 0$ and $\phi'_k(t) > 0$

- $A_k(t)$and $\phi'_k(t)$ are not constants. Thus, the momentary behavior of the oscillation cannot be captured by the Fourier transform.
- We use Time-Frequency representations to analyze this type of signals.

Limits

Limits

Main limits of time-frequency analysis :

- Heisenberg uncertainty principle,
- Interference between modes.

Different methods have been developed to alleviate the shortage of these analyses (EMD, Reassignment, Synchrosqueezing transform...). But there is still a lack : they are not efficient (particularly Synchrosqueezing) to represent signals with fast varying frequency. We constructed a time frequency representation which has the property to represent this type of signals.

Generalized intrinsic mode type function

Generalized intrinsic mode type function $\mathcal{Q}_\epsilon^{c_1,c_2,c_3}$

Fix constants $0 \leq \epsilon \ll 1, c2 > c1 > \epsilon$ and $c2 > c3 > \epsilon$. Consider the functional set $\mathcal{Q}_\epsilon^{c_1,c_2,c_3}$, which consists of functions in $C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with the following format :

$$g(t) = A(t)\cos(2\pi\phi(t))$$

which satisfy : $A \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\phi \in C^3(\mathbb{R})$, where $\forall t \in \mathbb{R}$:

1. $c1 \leq A(t) \leq c2$,
2. $c1 \leq \phi'(t) \leq c2$,
3. $|\phi''(t)| \leq c3$,

and the growth condition, $\forall t \in \mathbb{R}$:

1. $|A'(t)| \leq \epsilon\phi'(t)$,
2. $|\phi'''(t)| \leq \epsilon\phi'(t)$.

Adaptive harmonic Model

Adaptive harmonic model $\mathcal{Q}_{\epsilon,d}^{c_1,c_2,c_3}$

$$g(t) = \sum_{k=1}^{K} A_k(t)\cos(2\pi\phi_k(t))$$

where $A_k(t)\cos(2\pi\phi_k(t)) \in \mathcal{Q}_{\epsilon}^{c_1,c_2,c_3}$ and $\phi'_{k+1}(t) - \phi'_k(t) > d$

How to get the time-frequency representation of this type of functions ?

We would like to construct a convex functional. Therefore, we need to find two terms :

- The data fidelity term,
- The regularization term.

The data fidelity term correspond to the relation between the measured signal and the constructed transform. This is the reconstruction formula.

Data fidelity term

To find it, let consider the ideal time-frequency representation(iTFR), denoted as $R_f(t,\omega)$.
We would expect to have $R_f(t,\omega)$., satisfying

$$R_f(t,\omega) = \sum_{k=1}^{K} A_k(t) e^{i2\pi\phi_k(t)} \delta_{\phi'_k(t)}(\omega).$$

Data fidelity term

We can easily find that

$$f(t) = \Re \int R_f(t,\omega) d\omega$$

Regularization term

Differentiation properties

- For signal with slow varying frequency, i.e $|\phi''(t)| \leq \epsilon|\phi'(t)|, \forall t \in \mathbb{R}$, one can prove that :

$$\partial_t R_f(t,\omega) \approx i2\pi\omega R_f(t,\omega).$$

- When $f \in \mathcal{Q}_\epsilon^{c_1,c_2,c_3}$:

$$\partial_t R_f(t,\omega) \approx i2\pi\omega R_f(t,\omega) - \phi''(t)\partial_\omega R_f(t,\omega).$$

Regularization term

$$\int |\partial_t F(t,\omega) - i2\pi\omega F(t,\omega) + \alpha(t)\partial_\omega F(t,\omega)|^2 dt d\omega$$

where $\alpha(t)$ is used to capture the chirp factor associated with the "fast varying instantaneous frequency".

Functional

Functional to minimize

$$\mathcal{H}(F,\alpha) = \int_{\mathbb{R}} \left| \Re \int_{\mathbb{R}} F(t,\omega)d\omega - f(t) \right|^2 dt$$
$$+ \mu \iint_{\mathbb{R}} |\partial_t F(t,\omega) - i2\pi\omega F(t,\omega) + \alpha(t)\partial_\omega F(t,\omega)|^2 \, dtd\omega$$
$$+ \lambda \|F\|_{L^1} + \gamma \|\alpha\|_{L^2}^2$$

First, for a numerical optimization we need to discretize it.

Numerical discretization

- Due to the data collection procedure, we get $\boldsymbol{F_{nm}} = F(t_m, \omega_n)$, $\boldsymbol{\alpha_m} = \alpha(t_m)$
- We discretize $\mathcal{H}(F,\alpha)$ directly by the rectangle method.

Strategy

Alternate optimization

$$\begin{cases} \boldsymbol{F_{k+1}} = \arg\min_{\boldsymbol{F}} \mathcal{H}(\boldsymbol{F}, \boldsymbol{\alpha_k}) \\ \boldsymbol{\alpha_{k+1}} = \arg\min_{\boldsymbol{\alpha}} \mathcal{H}(\boldsymbol{F_{k+1}}, \boldsymbol{\alpha}) \end{cases}$$

- Over $\boldsymbol{\alpha}$, we can reach the global minimizer of $\boldsymbol{\alpha} \mapsto \mathcal{H}(\boldsymbol{F_{k+1}}, \boldsymbol{\alpha})$,
- To find a minimizer of $\boldsymbol{F} \mapsto \mathcal{H}(\boldsymbol{F}, \boldsymbol{\alpha_k})$, the use of an iterative algorithm is required.

Minimization of $\mathcal{H}_\alpha = \mathcal{H}(\cdot, \alpha)$

$\mathcal{H}_\alpha$ can be decomposed as

$$\mathcal{H}_\alpha(\boldsymbol{F}) = \mathcal{G}_\alpha(\boldsymbol{F}) + \Psi_\alpha(\boldsymbol{F})$$

where $\mathcal{G}_\alpha$ is a convex and Lipschitz differentiable function, and Ψ_α a convex but non-smooth function.

FISTA

Under these conditions, the Fast Iterative Shrinkage/Thresholding Algorithm (FISTA) can then be employed. FISTA has the great advantage to reach the optimal rate of convergence; that is, if $\check{F}$ is the convergence point, $\mathcal{H}_\alpha(F_k) - \mathcal{H}_\alpha(\check{F}) = \mathcal{O}\left(\frac{1}{k^2}\right)$

FISTA algorithm

Algorithm 1 FISTA algorithm for $\mathcal{H}_\alpha : F = \text{FISTA}(F_0, \alpha, \epsilon)$.

The initial values are $F_0 \in \mathbb{C}^{(N+1)\times(M+1)}$, $z_0 = F_0$.

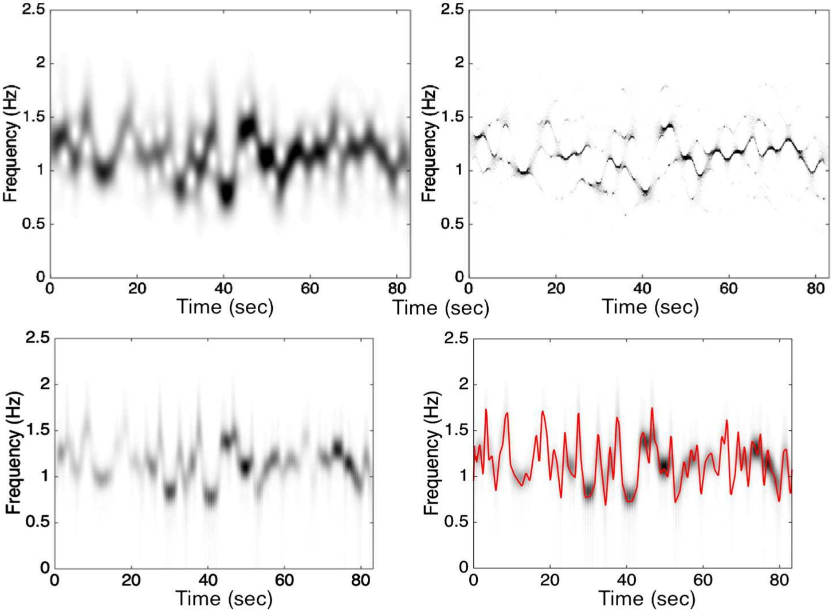
while stopping criterion is false **do**

- Gradient step : $F_{k+1/2} \leftarrow z_k - \frac{1}{L}\nabla\mathcal{G}_\alpha|_{z_k}$;
- Proximal step : $F_{k+1/2} \leftarrow F_{k+1/2}\left(1 - \frac{\lambda/L}{|F_{k+1/2}|}\right)^+$;
- Monotonic step : Control that $\mathcal{H}(F_{k+1/2}, \alpha) < \mathcal{H}(F_k, \alpha)$
- Relaxation step : $z_{k+1} \leftarrow F_{k+1} + \frac{k}{k+2}(F_{k+1} - F_k) + \frac{k+1}{k+2}(F_{k+1/2} - F_k)$;

$k = k + 1$;

end while

Output F.


2.5
2
1.5
1
0.5
0
Frequency (Hz)
0
20
40
60
80
Time (sec)
Time (sec)
Time (sec)

Some numerical results

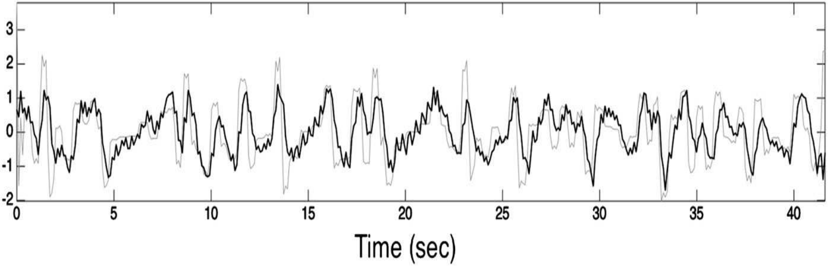
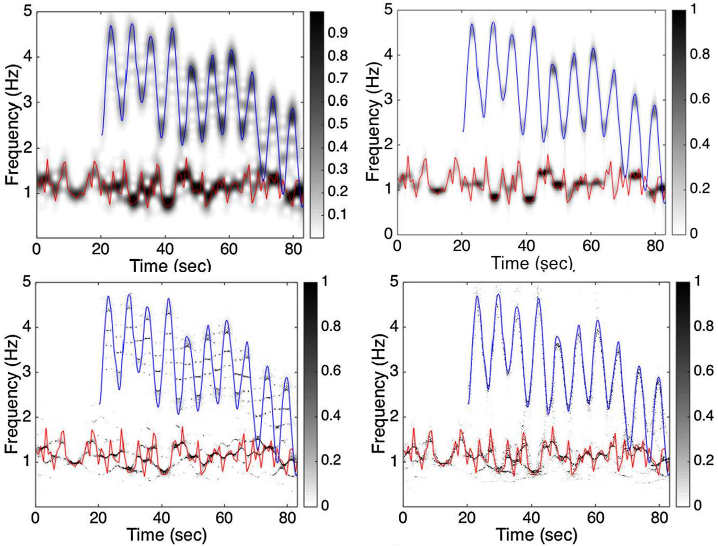


Figure – Chirp factor

Some results



Conclusion

- While with the help of FISTA the optimization process can be carried out, it is still not numerically efficient enough for practical usage.
- When there are more than one oscillatory component, we could improve the result.
- How to choose an optimal set of parameters μ,λ and γ?
- Theoretically studying the noise influence on the algorithm is important.